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Dynamic Mean Preserving Spreads

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Abstract

We extend the celebrated Rothschild and Stiglitz (1970) definition of Mean Preserving Spreads to scalar diffusion processes. We provide sufficient conditions under which a family of diffusion processes satisfies the dynamic counterparts to the famous Rothschild and Stiglitz integral conditions. We prove that the only Brownian bridge with non-constant drift that displays the Dynamic Mean-Preserving Spread (DMPS) property is given by the ballistic super-diffusive process. We illustrate our results in the context of the cannonical examples of investment under uncertainty and option pricing.

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1 Introduction

In their celebrated paper, Rothschild and Stiglitz (1970) defined the concept of second-order stochastic dominance, often referred to as a Mean-Preserving Spread (MPS).[1] This concept has become a workhorse of microeconomic analysis, with applications ranging from finance to the study of inequality (see, e.g. the standard textbooks by Laffont (1990) and Gollier (2001)). In this paper, we provide the dynamic counterpart to MPSs in the context of scalar diffusion processes. A remarkable result is that, for a broad class of standard diffusion processes (Brownian bridges), a specific functional form which corresponds to super-diffusive ballistic noise constitutes the *sole* process with non-constant drift that displays the dynamic MPS (DMPS) property. While the functional form is non-normal, its properties allow for simple closed-form results in a broad range of applications.

This paper is organized as follows. In Section 2, we derive our main result that provides sufficiency conditions –essentially an antisymmetry condition on the derivative with respect to risk of the Radon-Nicodym derivative associated with the transition probability density– for a family of risk-parameterized scalar diffusion processes

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[1]Various terminologies apply. For example, Chateauneuf, Cohen, and Meilijson (2004) refer, more accurately, to a mean-preserving increase in risk (MPIR).

to display the DMPS property. Using a result due to Benjamini and Lee (1997), we then prove, for Brownian bridges, that the ballistic super-diffusive process given by $dX_t = \left\{\sqrt{2\lambda}\tanh\left[\sqrt{2\lambda}X_t\right]\right\}dt + dW_t$ where W_t is the standard Brownian Motion, is the *only* process with non-constant drift that displays the DMPS property. As first noted by Rogers and Pitman (1981), it turns out that this process has an extremely simple representation in terms of the superposition of two drifted Wiener processes, leading to particularly simple closed-form solutions in common applications. Section 3 shows that the DMPS property is **not** the same thing as an increase in σ in a diffusion process of the form $dX_t = b(x)dt + \sigma dW_t$. We do this by (i) studying the curvature of the time-invariant probability measure for scalar processes and (ii) showing that, contrary to an increase in σ, a DMPS may violate the Certainty Equivalence Principle. Section 4 provides an illustration of our results by applying them to the cannonical problems of investment under uncertainty and option pricing.

2 The main result

On $\mathbb{R}$, consider the scalar diffusion process (DP) X_t defined by the stochastic differential equation (SDE):

$$\begin{cases} dX_t = b(X_t)dt + \sigma dW_t, \\ X_0 = x_0, \end{cases} \tag{1}$$

where W_t is the standard Brownian Motion (BM). Associated with equation (1), we define, for any function $\varphi(x) : \mathbb{R} \to \mathbb{R}^+$, the diffusion operator $\mathcal{L}$:

$$\mathcal{L}_x\varphi(x) := \left[\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}\right]\varphi(x). \tag{2}$$

For time $s \le t$ let us write the transition probability density (TPD) which describes the DP of equation (1), as $q(s, x_0|x, t)$. Assume that $h(x,t)$ is a positive definite solution of:

$$\frac{\partial}{\partial t}H(x,t) + \mathcal{L}_x\left[H(x,t)\right] = 0. \tag{3}$$

Then applying Itô's Lemma to $H(X_t, t)$ with the process X_t defined by (1), we have $\mathbb{E}\left\{\frac{d}{dt}H(X_t,t) = 0\right\}$, where $\mathbb{E}(\cdot)$ stands for the expectation operator. By Theorem 2.1 of Dai Pra (1991), we can write a dual process:

$$\begin{cases} d\hat{X}_t = \left\{b(\hat{X}_t) + \sigma^2\frac{\partial}{\partial x}\log\left[H(x,t)\right]|_{x=\hat{X}_t}\right\}dt + \sigma dW_t, \\ \hat{X}_0 = x_0. \end{cases} \tag{4}$$

The TPD $Q(s, x_0|x, t)$ characterizing the dual diffusion $\hat{X}_t$ reads:

$$Q(s, x_0|x, t) = \left[\frac{H(x, t)}{H(x_0, s)]}\right] q(s, x_0|x, t). \tag{5}$$

The function $z_t := \left[\frac{H(x,t)}{H(x_0,s)]}\right]$ is the Radon-Nicodym derivative relating the TPD's $Q(s, x_0|x, t)$ with $q(s, x_0|x, t)$ and the process $Z_t := \left[\frac{H(X_t,t)}{H(X_0,s)]}\right]$ is a martingale with $\mathbb{E}[Z_t] = 1$.

Consider the class of positive definite functions

$$H(x, t) = e^{-\lambda t} h^{(\lambda)}(x), \qquad h^{(\lambda)}(x) \geq 0, \qquad x \in \mathbb{R}, \tag{6}$$

which in view of equation (3) implies:

$$\mathcal{L}_x \left[h^{(\lambda)}(x)\right] = \lambda\, h^{(\lambda)}(x), \tag{7}$$

where λ is a positive constant, which will correspond in what follows to the Rothschild and Stiglitz parameter of increasing risk. Substituting equation (6) into equation (4), we can write a λ-family of TPDs as:

$$Q^{(\lambda)}(0, x_0|x, t) = e^{-\lambda t} \left[\frac{h^{(\lambda)}(x)}{h^{(\lambda)}(x_0))}\right] q(s, x_0|x, t). \tag{8}$$

As Z_t is a martingale with $\mathbb{E}[Z_t] = 1$, equation (8) defines itself a normalized TPD. With $x_0 = 0$ and $s = 0$, the mean of $\hat{X}_t$ is given by:

$$m^\lambda(t) = \mathbb{E}^\lambda \left\{\hat{X}_t\right\} = \frac{e^{-\lambda t}}{h^\lambda(0)} \int_{\mathbb{R}} x h^\lambda(x) q(0, 0|x, t) dx. \tag{9}$$

Let us now assume that, in equation (1), we have $b(x) = -b(-x)$. In view of equation (2), this implies symmetry: $\mathcal{L}_x \equiv \mathcal{L}_{-x}$. In turn, equation (7) implies that $q(0, 0|x, t) = q(0, 0| - x, t)$ and $h^\lambda(x) = h^\lambda(-x)$, and therefore the antisymmetry of the integrand in equation (9). It follows that $m^\lambda(t) \equiv 0$.

A Dynamic Mean-Preserving Spread (or dynamic mean-preserving increase in risk) with respect to the parameter of increasing risk λ is defined by the usual two integral conditions:

Definition 1 *(Rothschild and Stiglitz, 1970): Let* $\mathcal{P}^{(\lambda)}(x, t) := \int_{-\infty}^{x} Q^{(\lambda)}(0, 0|y, t) dy$. *Then a Dynamic Mean-Preserving Spread (DMPS) is defined by:*

$$\begin{cases} i) & \frac{\partial}{\partial \lambda}\left[\int_{\mathbb{R}} \mathcal{P}^{(\lambda)}(x, t) dx\right] = 0, \\ ii) & \frac{\partial}{\partial \lambda}\left[\int_{-\infty}^{x} \mathcal{P}^{(\lambda)}(y, t) dy\right] \geq 0. \end{cases} \tag{10}$$

The integral conditions in Definition 1 are essentially identical to the integral conditions in Rothschild and Stiglitz (1970) except that instead of a cumulative density as the integrands we now have a TPD which is also a function of t. We then have the following Proposition:

Proposition 1 *Let $R^{(\lambda)}(x) := \frac{\partial}{\partial\lambda}h^{(\lambda)}(x)$. A sufficient condition for the integral conditions (10) to be satisfied is:*

$$R^{(\lambda)}(x) = R^{(\lambda)}(-x) \geq 0. \tag{11}$$

Proof. Substituting (11) into (10), we can write:

$$\begin{cases} a) \quad \frac{\partial}{\partial\lambda}\left[\int_{\mathbb{R}} \mathcal{P}^{(\lambda)}(x,t)dx\right] = \frac{e^{-\lambda t}}{h^{\lambda}(0)}\int_{\mathbb{R}} \underbrace{\left\{\int_{-\infty}^{x} \left[R^{(\lambda)}(y)q(0,0|y,t)\right]dy\right\}}_{\Psi^{(\lambda)}(x)} dx, \\ b) \quad \Phi^{(\lambda)}(x) =: \frac{\partial}{\partial\lambda}\left[\int_{-\infty}^{x} \mathcal{P}^{(\lambda)}(y,t)dy\right] = \frac{e^{-\lambda t}}{h^{\lambda}(0)}\int_{-\infty}^{x} \Psi^{(\lambda)}(y)dy. \end{cases} \tag{12}$$

From the condition given by equation (11) and the fact that $q(0,0|x,t) = q(0,0|-x,t)$, we conclude that $\Psi^{(\lambda)}(x) = -\Psi^{(\lambda)}(-x)$ and its integral over $\mathbb{R}$ vanishes, leading to the fulfillment of the first integral condition. Now consider the curvature $\rho^{(\lambda)}(x)$ of $\Phi^{(\lambda)}(x)$, which reads:

$$\rho^{(\lambda)}(x) = \frac{\partial^2}{\partial x^2}\left[\Phi^{(\lambda)}(x)\right] = \left[R^{(\lambda)}(x)q(0,0|x,t)\right] \geq 0. \tag{13}$$

As from part $a)$ of equation (12) we know that $\Phi^{(\lambda)}(\infty) = 0$, this can only be achieved provided we have $\Phi^{(\lambda)}(x) \geq 0$. The second integral condition is therefore verified. ■

In order to illustrate the workings of Proposition 1, we now consider two examples of $\lambda-$family diffusion processes that satisfy the antisymmetry and positivity properties of Proposition 1 and therefore display the DMPS property. The second example will be of particular interest in what follows.

Example 1 HERMITE FUNCTIONS. *Let us consider equation (1) with $b(x) = x$, $\sigma = \sqrt{2}$. This special choice leads to the Hermite function differential operator:*

$$\mathcal{L}(\cdot) = \left[\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x}\right](\cdot). \tag{14}$$

Let:

$$h^{(\lambda)}(x) = e^{-\frac{x^2}{4}}\mathcal{Y}(\lambda, x). \tag{15}$$

Equations (7) and (14) then lead to:[2]

$$\frac{d^2}{dx^2}\left\{\mathcal{Y}(\lambda,x)\right\} = \left[\frac{x^2}{4} + \left(\lambda - \frac{1}{2}\right)\right]\mathcal{Y}(\lambda,x), \quad \lambda \geq 0, \tag{16}$$

[2]See 9.255/1 of Gradshteyn and Ryzhik (1994).

which is solved using Hermite functions,[3] *expressible in terms of confluent hypergeometric functions:*

$$\mathcal{Y}(\lambda, x) := e^{-\frac{x^2}{4}}\, {}_1F_1\left[\frac{\lambda}{2}, \frac{1}{2}; \frac{x^2}{2}\right] \geq 0. \tag{17}$$

Let us now calculate:

$$R^{(\lambda)}(x) = \frac{d}{d\lambda}\mathcal{Y}(\lambda, x) = e^{-\frac{x^2}{4}} \frac{d}{d\lambda}\left\{ {}_1F_1\left[\frac{\lambda}{2}, \frac{1}{2}; \frac{x^2}{2}\right]\right\}, \tag{18}$$

where it is obvious that the antisymmetry portion of Proposition 1 is satisfied. Note that:

$$\begin{aligned} \frac{d}{d\lambda}\left\{ {}_1F_1\left[\frac{\lambda}{2}, \frac{1}{2}; \frac{x^2}{2}\right]\right\} &= \frac{1}{2}\frac{d}{d(\lambda/2)}\left\{ {}_1F_1\left[\frac{\lambda}{2}, \frac{1}{2}; \frac{x^2}{2}\right]\right\} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)}\left[\Psi(\frac{\lambda}{2}+n) - \Psi(\frac{\lambda}{2})\right]\frac{x^n}{n!}, \end{aligned}$$

where $\Psi(z)$ is the digamma function. Since $\Psi(n + \frac{\lambda}{2}) = \sum_{k=0}^{n-1}\frac{1}{k+\frac{\lambda}{2}} + \Psi(\frac{\lambda}{2})$ (see equation 6.3.6 in Abramowitz and Stegun (1964)), it follows that:

$$R^{(\lambda)}(x) = \frac{1}{2}e^{-\frac{x^2}{4}}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)}\frac{x^{2n}}{2^n n!}\left\{\sum_{k=1}^{n-1}\frac{1}{k+(\lambda/2)}\right\} \geq 0.$$

Example 2 BALLISTIC SUPER-DIFFUSIVE PROCESSES. *Let us now apply Proposition 1 to the pure Brownian motion obtained when $b(x) = 0$ and $\sigma = 1$ in equation (1). Accordingly, $\mathcal{L}_x(\cdot) = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\cdot)$ and equation (7) and its positive class of solutions read:*

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}h^{(\lambda)}(x) = \lambda h^{(\lambda)}(x) \quad \Rightarrow \quad h^{(\lambda)}(x) = A\cosh\left(\sqrt{2\lambda}x\right), \tag{19}$$

where A is an arbitrary constant. Here we have $R^{(\lambda)}(x) = A\sinh\left(\sqrt{2\lambda}x\right)$ and equation (11) reads:

$$R^{(\lambda)}(x) = \frac{d}{d\lambda}A\cosh\left(\sqrt{2\lambda}x\right) = \frac{Ax}{2\sqrt{2\lambda}}\sinh\left[\sqrt{2\lambda}x\right] = R^{(\lambda)}(-x). \tag{20}$$

In view of equation (4), we conclude that the resulting process reads:

$$dX_t = \left\{\sqrt{2\lambda}\tanh\left[\sqrt{2\lambda}X_t\right]\right\}dt + dW_t, \tag{21}$$

[3] Also known as Weber parabolic cylinder functions. See sections 9.24 and 9.25 of Gradshteyn and Ryzhik (1994).

and has the TPD:

$$Q(0,0|x,t) = \frac{e^{-\lambda t}}{\sqrt{2\pi t}} \cosh\left[\sqrt{2\lambda} x\right] e^{-\frac{x^2}{2t}} = \frac{1}{2\sqrt{2\pi t}} \left\{ e^{-\frac{(x-\sqrt{2\lambda}t)^2}{2t}} + e^{-\frac{(x+\sqrt{2\lambda}t)^2}{2t}} \right\}. \tag{22}$$

Note that the right-hand expression of equation (22) shows that the process is a superposition of two $\pm\sqrt{2\lambda}$-drifted Wiener processes. This implies that equation (21) can be rewritten as:

$$dX_t = \mathcal{B} dt + dW_t, \tag{23}$$

where $\mathcal{B}$ stands for a Bernoulli random variable taking the values $\pm\sqrt{2\lambda}$. This property has previously been noted by Rogers and Pitman (1981).[4] Our Example 2 is of considerable importance in applications, as we shall illustrate in Section 4. This is because we have the following Proposition:

Proposition 2 *(Benjamini and Lee, 1997) For any diffusion process which is a Brownian bridge, the* ***only*** *diffusion process with non-constant drift $\mu(X_t)$ which satisfies the DMPS property is that given by equation (21): $dX_t = \left\{\sqrt{2\lambda}\tanh\left[\sqrt{2\lambda}X_t\right]\right\} dt + dW_t$.*

Proof. The Proposition follows directly from Theorem 1 in Benjamini and Lee (1997). ■

In the context of economics, restricting attention to Brownian bridges is not particularly stringent in that this means that X_t is pinned down at both of its endpoints. Of course, if one considers a conditioned diffusion process and restricts attention to constant drifts $\mu = k$, it is well known that the resulting process is a Brownian bridge. The Theorem of Benjamini and Lee (1997) applied in Proposition 2 states that if the process is a Brownian bridge (subject to bounded and continuous first and second derivatives for $\mu(X_t)$) then either the drift is constant, or it takes the form given in Proposition 2.

The ballistic super-diffusive process of Example 2 and Proposition 2 has been previously considered in Hongler (1979) and Hongler, Filliger, and Blanchard (2006). An illustration of $\mathcal{P}^\lambda(x,t) := \int_{-\infty}^{x} Q^\lambda(0,0|y,t)dy$ where $Q^\lambda(0,0|y,t)$ is given by equation (22), for $\lambda = 1, 2, 5, 10$ is provided in Figure 1. Consider the $\mathcal{P}^\lambda(x,t)$ surfaces for $\lambda = 1$ and $\lambda = 2$. For a given t, these would correspond to two cumulative densities in the standard Rothschild and Stiglitz graphical illustration, with the former having lower risk than the latter. It is visually obvious that the area corresponding to the vertical distance between the two curves to the right of $x = 0$ (the mean) and to the left of $x = 0$ (note that x runs from negative to positive from right to left) are equal: this corresponds to the first integral condition in Definition 1. If one were to "stop" at some positive value y of x

[4]See *their* example 2.

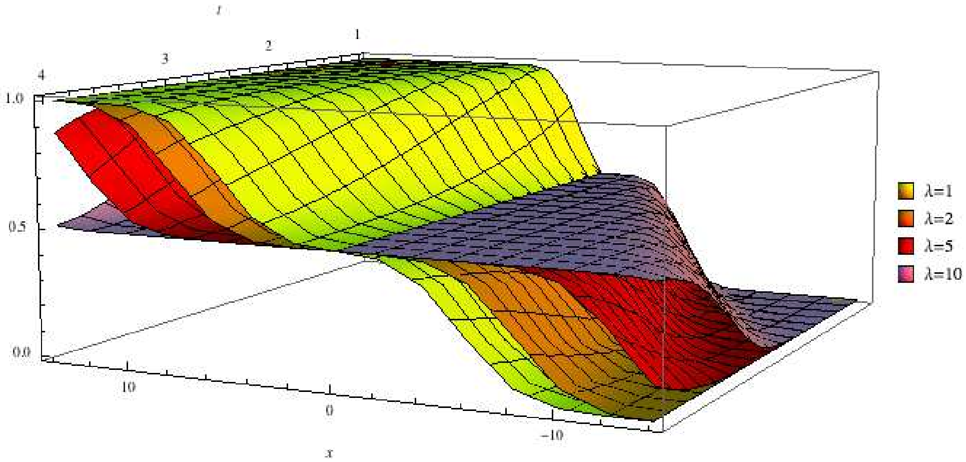

Figure 1: An illustration of $\mathcal{P}^{\lambda}(x,t) := \int_{-\infty}^{x} \frac{1}{2\sqrt{2\pi t}} \left\{ e^{-\frac{(y-\sqrt{2\lambda}t)^2}{2t}} + e^{-\frac{(y+\sqrt{2\lambda}t)^2}{2t}} \right\} dy$, for $\lambda = 1, 2, 5, 10$.

to the left of $x = 0$, the total "positive" distance between the two curves over $(-\infty, 0)$ would of necessity outweight the "negative" distance between the two curves over $(0, y)$: this corresponds to the second integral condition of Definition 1.

3 A DMPS is not the same thing as an increase in σ

3.1 The curvature of the time-invariant probability measure

To show that using a DMPS as the driving noise in diffusion processes leads to behavior that is drastically different from that stemming from an increase in the variance term σ in front of the White Gaussian Noise (WGN), consider the behavior of the time-invariant probability measure for scalar processes. Consider the process:

$$dX_t = b(x)dt + \sigma dW_t. \tag{24}$$

The forward Fokker-Planck equation (FPE) for the TPD $P(x, t|x_0, 0)$ associated with equation (24) is:

$$\frac{\partial}{\partial t} P(x, t|x_0, 0) = -\frac{\partial}{\partial x} \left[b(x) P(x, t|x_0, 0) \right] + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} P(x, t|x_0, 0), \tag{25}$$

and the time-invariant (or stationary) measure $P_s(x) = \lim_{t\to\infty} P(x, t|x_0, 0)$ is obtained by solving equation (25) with vanishing left-hand side. Formally integrating twice with respect to x (with vanishing constants of integration), we obtain:

$$0 = -\left[b(x) P_s(x) \right] + \frac{\sigma^2}{2} \frac{\partial}{\partial x} P_s(x) \;\Rightarrow\; P_s(x) = \mathcal{N} e^{\frac{2}{\sigma^2} B(x)}, \tag{26}$$

where $B(x) = \int^x b(\xi)d\xi$ and $\mathcal{N}$ is a normalization factor which exists for globally attracting drifts, (i.e. $\lim_{|x|\to\infty} B(x) = -\infty$). From equation (26), it is clear that increasing the variance σ^2 spreads the $P_s(x)$ without affecting the positions of its extrema.

Let us now consider the impact of using a DMPS process in equation (24). To this aim, we write:

$$\begin{cases} dX_t = b(x)dt + \sigma dZ_t, & X_0 = x_0, \\ dZ_t = \sqrt{2\lambda}\tanh\left[\sqrt{2\lambda}Z_t\right]dt + dW_t & Z_0 = 0. \end{cases} \tag{27}$$

The degenerate diffusion process in $\mathbb{R}^2$ can be rewritten in vector form as:

$$\begin{cases} d\vec{Y}_t = \vec{F}(\vec{Y}_t)dt + \mathbf{\Sigma} d\vec{W}_t \\ \vec{F}(\vec{Y}_t) = \left[b(X_t)dt + \sqrt{2\lambda}\tanh\left[\sqrt{2\lambda}Z_t\right], \sqrt{2\lambda}\tanh\left[\sqrt{2\lambda}Z_t\right]\right], \\ \mathbf{\Sigma} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \end{cases} \tag{28}$$

and is characterized by the PDF $P(x, z, t|x_0, 0, 0)$ solving the FPE:

$$\begin{cases} \frac{\partial}{\partial t}P(x,t|x_0,0) = \mathcal{F}\frac{\partial}{\partial t}P(x,t|x_0,0), \\ \mathcal{F}(\cdot) = \left\{\begin{bmatrix} -\frac{\partial}{\partial x}\left(b(x) + \sigma\tanh\left[\sqrt{2\lambda}z\right]\right) \\ -\frac{\partial}{\partial z}\left(+\sigma\tanh\left[\sqrt{2\lambda}z\right]\right) \end{bmatrix}\right\}(\cdot) + \overbrace{\left\{\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x\partial z} + \frac{1}{2}\frac{\partial^2}{\partial z^2}\right\}}^{=(\partial x,\partial z)\mathbf{\Sigma}\,\mathbf{\Sigma}^T(\partial x,\partial z)^T}(\cdot). \end{cases} \tag{29}$$

The stationary measure $P_s(x, z)$ in $\mathbb{R}^2$ solves $\mathcal{F}(P_s(x, z)) = 0$. For arbitrary $b(s)$, $P_s(x, z)$, $\mathcal{F}(P_s(x, z)) = 0$ cannot generally be integrated in closed form. However, in view of equation (27), we are mostly interested in the marginal measure $P_{sM}(x)$ resulting from the z-integration $P_{sM}(x) = \int_{\mathbb{R}} P_s(x, z)dz$. Using the representation equation (23), it is however easy to calculate $P_{sM}(x)$. To see why, one uses the representation (23) to rewrite equation (27) as the diffusion process:

$$dX_t = [b(x)dt + \mathcal{B}]\,dt + \sigma dW_t, \qquad X_0 = x_0, \tag{30}$$

leading to the marginal stationary measure:

$$P_{sM}(x) = \mathcal{N}\left[e^{\frac{2B^+(x)}{\sigma^2}} + e^{\frac{2B^-(x)}{\sigma^2}}\right] = \mathcal{N}\cosh(\sqrt{2\lambda}x)e^{\frac{2B(x)}{\sigma^2}}, \tag{31}$$

where we have defined: $B^{\pm}(x) = B(x) \pm \sqrt{2\lambda}x$. Comparing equation (26) and equation (31), it is obvious that driving the system with the DMPS process is drastically different from simply modifying the variance. In particular when $b(x) = -b(-x)$, leading to $P_{sM}(x) = P_{sM}(-x)$, let us calculate the curvature at the origin $\rho_{(\sigma,\lambda)} = \left[\frac{\partial^2}{\partial x^2}P_{sM}(x)\right]|_{x=0}$.

We have:

$$\rho_{(\sigma,\lambda)} = \underbrace{\frac{2}{\sigma^2}\left[\frac{\partial}{\partial x}b(x)\right]|_{x=0}}_{:=\rho_{(\sigma,0)}} + 2\lambda = \rho_{(\sigma,0)} + 2\lambda, \tag{32}$$

where $\rho_{(\sigma,0)}$ is the curvature of the probability measure induced by the WGN in equation (26). This shows that the behavior obtained by driving the system with the DMPS cannot be derived from a simple change of the variance in front of the WGN. In particular, we may well have $\rho_{(\sigma,0)} < 0$ but $\rho_{(\sigma,\lambda)} > 0$ showing that the DMPS induces a change in the number of modes of $P_{sM}(x)$ (i.e. odd number of modes when $\rho_{(\sigma,0)} < 0$, even when $\rho_{(\sigma,0)} > 0$).

3.2 A DMPS may violate the Certainty Equivalence Principle

As in section 2.6 of Karatzas (1996) consider, for time $t \in [0, T]$, the controlled scalar stochastic process in finite time horizon defined by the diffusion process:

$$dX_t = \pi(t)\left[bdt + \sigma dW_t\right], \qquad X_0 = x \in [0, 1], \tag{33}$$

where b is a constant drift, dW_t is the standard Wiener Process and $\pi(t)$ is a control. We consider the class $\mathcal{H}(x)$ of admissible controls $\pi(t)$ which are progressively measurable and for which we have:

$$\int_0^T \pi^2(s)ds < \infty \qquad \text{and} \quad 0 \le X_t \le 1 \ \text{ for } t \in [0, T]. \tag{34}$$

Theorem 2.6.4 of Karatzas (1996) computes the explicit value function $G(x)$ of the problem:

$$G(x) := \sup_{\pi(\cdot)\in\mathcal{H}(x)} \text{Prob}\left\{X^{x,\pi}(T) = 1\right\}, \tag{35}$$

and provides the optimal process $\hat{\pi}(t)$ that attains the supremum in equation (35). In words, the goal is to determine the optimal control $\hat{\pi}(t)$ that maximizes, over the finite time horizon T, the probability of reaching the right-hand boundary 1 without touching the left-hand boundary 0.

Now consider the same problem when the noise source dW_t is replaced by the DMPS process of Proposition 2. Using the Bernoulli representation of equation (23), we can rewrite equation (33) as:

$$\begin{cases} dX_t = \pi(t)\left[bdt + dZ_t\right], \qquad X_0 = x \in [0, 1], \\ dZ_t = \left[\sqrt{2\lambda}\tanh\left(\sqrt{2\lambda}Z_t\right)\right]dt + dW_t \end{cases} \tag{36}$$

Again using (23), we can rewrite equation (36) as:

$$dX_t = \pi(t)\left[\hat{b}dt + dZ_t\right], \qquad X_0 = x \in [0,1], \tag{37}$$

where $\hat{b}$ is a Bernoulli random variable with Prob $\left\{\hat{b} = b + \sqrt{2\lambda}\right\}$ = Prob $\left\{\hat{b} = b - \sqrt{2\lambda}\right\} = 1/2$ and hence $\hat{b}$ is drawn from the probability density function $p(x)dx$:

$$p_b(x)dx = \frac{1}{2}\left[\delta(x - b - \sqrt{2\lambda}) + \delta(x - b + \sqrt{2\lambda})\right] dx, \tag{38}$$

where $\delta(x - z)dx$ is the Dirac mass at z. Using a martingale approach, Karatzas (1997) establishes that, provided the support of $p_b(x)$ lies strictly in $\mathbb{R}^+$ or in $\mathbb{R}^-$, the optimal control in presence of a random drift $\hat{b}$ can be directly obtained from the deterministic case by a simple substitution $b \mapsto \hat{b}(t) = \mathbb{E}\left\{\hat{b} \mid \mathcal{F}(t)\right\}$, where $\hat{b}(t)$ is the conditional expectation of $\hat{b}$ given the observation of the process up to time t. This is called the Certainty Equivalence Principle (CEP). Conversely, for cases where the support $p_b(x)$ crosses the origin, the CEP is violated and the resulting optimal control is also explicitly calculated in Karatzas (1997). Clearly, in the deterministic situation of equation (33), one can write $p_b(x) = \delta(b - x)$ and, for $b \neq 0$ the support of $p_b(x)$ never crosses the origin. Then for all values of σ, the CEP holds. This is not the case for the DMPS process in equation (36) or equivalently (37). Here, when $\sqrt{2\lambda} > b$, the support of $p_b(x)$ is not entirely contained in $\mathbb{R}^+$ (or $\mathbb{R}^-$), and therefore the CEP does not hold. This clearly shows that a DMPS is not equivalent to a simple modification of the variance parameter σ lying in front of a WGN.

4 Illustrations

4.1 Optimal investment under uncertainty

Following Abel (1983) and Abel and Eberly (1994), we consider the dynamics given by:

$$\begin{cases} dK_t = [I_t - \delta K_t]\, dt, \\ d\epsilon_t = \mu\left(\epsilon_t + \sigma(\epsilon_t)\xi_t\right) dt, \end{cases} \tag{39}$$

where L_t stands for labor, K_t is the capital stock, δ is the depreciation rate, ξ_t is a stochastic process yet to be specified, ϵ_t stands for the ubiquitous randomness due to technology, fluctuating input and output prices and other random sources and $\sigma(\epsilon_t)$ is a multiplicative noise amplitude factor. Consider a risk-neutral firm which chooses, over an infinite time horizon, to maximize the expected present value of operating profit

$\pi(K_t, L_t, \epsilon_t)$:

$$V(K_t, \epsilon_{t+s}) = \max_{L_t} \int_0^\infty \mathbb{E}_t\{\pi(K_t, L_t, \epsilon_t)\} e^{-r(s-t)} ds, \tag{40}$$

where $r > 0$ stands for the discount rate and $\mathbb{E}_t\{\cdot\}$ stands for the expectation operator conditioned at time t. The corresponding Bellman equation is then given by:

$$rV(K_t, \epsilon_{t+s}) = \max_{L_t} \left\{ \pi(K_t, L_t, \epsilon_{t+s}) + \frac{1}{dt} \mathbb{E}\{dV_t(K_t, \epsilon_{t+s})\} \right\}. \tag{41}$$

Assuming that $\xi_t dt = dW_t$ where W_t is the Wiener process, the stochastic differential equation in equation (39) and Ito's Lemma yields a diffusion process that can be written as:

$$\begin{cases} rV(K_t, \epsilon_{t+s}) = \max_{L_t}\{\pi(K_t, L_t, \epsilon_{t+s}) + q_t(I_t - \delta K_t) + \mathcal{L}_\epsilon[V(K_t, \epsilon_{t+s})]\}, \\ \mathcal{L}_\epsilon[V(K_t, \epsilon_{t+s})] := \left[\mu(\epsilon)\frac{\partial}{\partial \epsilon} + \frac{\sigma^2(\epsilon)}{2}\frac{\partial^2}{\partial \epsilon^2}\right] V(K_t, \epsilon_{t+s}). \end{cases} \tag{42}$$

For the Cobb-Douglas functional form $\pi(K_t, L_t, \epsilon_t) = \max_{L_t}\{\epsilon_t L^\alpha K^{1-\alpha} - \omega L_t\} = h\epsilon^\theta K_t, \omega \geq 0$, $h := (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} \omega^{-\frac{\alpha}{1-\alpha}}$, the present value q_t^0 at time t of marginal profits accruing to capital is given by:

$$q_t^0 = h \int_0^\infty \mathbb{E}_t\left\{\epsilon_{(t+s)}^\theta\right\} e^{-(r+\delta)s} ds. \tag{43}$$

At this stage, it is clear from equation (43), that the remaining difficulty is to calculate the generalized moment $\mathbb{E}_t\left\{\epsilon_{(t+s)}^\theta\right\}$ for a general scalar diffusion process. Explicit results can be obtained when $\mu(\epsilon) = 0$ and $\sigma(\epsilon) = \sigma\epsilon$. In this case, the relevant stochastic differential equation in the Stratonovich interpretation reads $d\epsilon_t = \sigma\epsilon_t dW_t$ and, in the corresponding Itô interpretation, one has the drifted Wiener process $d\epsilon_t = \frac{1}{2}\epsilon_t\sigma^2 dt + \sigma\epsilon_t dW_t \quad \Rightarrow$ $d[\ln(\epsilon_t)] = \frac{1}{2}\sigma^2 dt + \sigma dW_t$, leading, for $y_t := \ln(\epsilon_t)$, to the log-normal probability density function:

$$P\{y, t+s \mid y_t, t\} dy = N\left(\left[y_t - \frac{1}{2}\sigma^2 s\right], \sigma^2 s; y\right) dy, \tag{44}$$

where $N(a, b; y) = \frac{1}{\sqrt{2\pi b}} e^{-\frac{(y-a)^2}{2b}}$. Equation (44) then leads to:

$$\mathbb{E}\left\{p_{(t+s)}^\theta\right\} = p_t^\theta \exp\left\{\frac{1}{2}\theta(\theta-1)\sigma^2 s\right\}, \tag{45}$$

from which it follows that:

$$q_t^0 = h \int_0^\infty p_t^\theta e^{\frac{1}{2}\theta(\theta-1)\sigma^2 s} e^{-(r+\delta)s} ds = \frac{h p_t^\theta}{r + \delta - \frac{1}{2}\theta(\theta-1)\sigma^2} \tag{46}$$

When $d\epsilon_t = \mu\epsilon_t + \sigma\epsilon_t dW_t$, we obtain:

$$q_t^{(\mu)} = h\int_0^{\infty} p_t^{\theta} e^{\mu s + \frac{1}{2}\theta(\theta-1)\sigma^2 s} e^{-(r+\delta)s} ds = \frac{h p_t^{\theta}}{r+\delta-\mu-\frac{1}{2}\theta(\theta-1)\sigma^2}. \tag{47}$$

Now assume that the stochastic productivity rate ϵ_t is driven by the ballistic noise source of Proposition 2 and replace (39) with:

$$\begin{cases} dK_t = [I_t - \delta K_t]\, dt, \\ d\epsilon_t = \epsilon_t\, \sigma\, d\xi_t, \\ d\xi_t = \mu \tanh(\mu\xi) dt + dW_t. \end{cases} \tag{48}$$

From equation (23), we know that we can write:

$$d\xi_t = \mathcal{B} dt + dW_t, \tag{49}$$

where $\mathcal{B} \in \{\{-\mu\}, \{+\mu\}\}$ is a Bernoulli random variable which is equal to $\pm\mu$ with probability $\frac{1}{2}$. In view of equation (49), we may directly use the result established in (47) for each of the two realizations of the Bernoulli process. This leads to:

$$\begin{aligned} q_t^{(\text{ballistic})} &= \frac{1}{2} h\, p_t^{\theta} \left\{ \frac{1}{r+\delta-\mu-\frac{1}{2}\theta(\theta-1)\sigma^2} + \frac{1}{r+\delta+\mu-\frac{1}{2}\theta(\theta-1)\sigma^2} \right\} \\ &= \frac{1}{2} h\, p_t^{\theta} \frac{(r+\delta)-\frac{1}{2}\theta(\theta-1)\sigma^2}{\left[(r+\delta)-\frac{1}{2}\theta(\theta-1)\sigma^2\right]^2 - \mu^2}. \end{aligned} \tag{50}$$

Note that (50) cannot be obtained from equation (47) by simply increasing the variance. This illustrates the fact that a DMPS of a probability density is not equivalent to an increase in the variance of the nominal density. For $\mu = 0$, we obviously recover equation (46). For small μ, equation (50) can be approximately rewritten as:

$$\begin{aligned} q_t^{(\text{ballistic})} &\simeq \frac{h\, p_t^{\theta}}{(r+\delta)-\frac{1}{2}\theta(\theta-1)\sigma^2} \left[1 + \frac{\mu^2}{\left[(r+\delta)-\frac{1}{2}\theta(\theta-1)\sigma^2\right]^2} \right] + \mathcal{O}(\mu^4) \\ &= q_t^0 \left[1 + \left[\frac{\mu\, q_t^0}{h\, p_t^{\theta}} \right]^2 \right] + \mathcal{O}(\mu^4). \end{aligned} \tag{51}$$

From equations (50) and (51), we conclude that, for a given p_t, the presence of ballistic noise leads to an increase in the marginal profits $q_t^{(\text{ballistic})}$ of currently installed capital.

4.2 Option pricing

Consider the Black-Scholes (BS) asset dynamics $X_t \geq 0$ driven by the DMPS Z_t, namely:

$$\begin{cases} dX_t = \{\mu\, dt + \sigma dZ_t\}\, X_t, \\ dZ_t = \sqrt{2\lambda} \tanh\left(\sqrt{2\lambda} Z_t\right) dt + dW_t, \\ X_0 = x_0 \qquad \text{and} \qquad Z_0 = 0. \end{cases} \tag{52}$$

Defining as usual $Y_t = \log(X_t)$ the process Y_t, Z_t is a diffusion process on $\mathbb{R}^+ \times \mathbb{R}$ and the associated Fokker-Planck equation (the underlying stochastic integrals are interpreted in the Itô sense) reads:

$$\begin{aligned}\partial_t P(y,z,t \mid x_0,0,0) = &-\partial_y \left[\mu + \sigma\sqrt{2\lambda}\tanh\left(\sqrt{2\lambda}\, Z_t\right) P(y,z,t \mid x_0,0,0)\right] \\ &-\partial_z \left[\sigma\sqrt{2\lambda}\tanh\left(\sqrt{2\lambda}\, Z_t\right) P(y,z,t \mid x_0,0,0)\right] \\ &+\tfrac{\sigma^2}{2}\partial_{yy}P(y,z,t \mid x_0,0,0) + \sigma\partial_{yz}P(y,z,t \mid x_0,0,0) \\ &+\tfrac{1}{2}\partial_{zz}P(y,z,t \mid x_0,0,0).\end{aligned} \tag{53}$$

Writing $P(y,z,t \mid y_0,0,0) = e^{-\lambda t}\cosh\left[\sqrt{2\lambda}z\right] Q(\hat{y},z,t \mid y_0,0,0)$, with the rescaling $\sigma\hat{y} := y - \mu t$, one immediately verifies that equation (53) reduces to a pure diffusion equation on $\mathbb{R} \times \mathbb{R}$:

$$\partial_t Q(y,z,t \mid y_0,0,0) = \left\{\tfrac{1}{2}\partial_{\hat{y}\hat{y}} + \partial_{\hat{y}z} + \tfrac{1}{2}\partial_{zz}\right\} Q(\hat{y},z,t \mid y_0,0,0), \tag{54}$$

which admits a Gaussian bivariate TPD with canonical structure:

$$\begin{cases} Q(y,z,t \mid y_0,0,0) = \frac{1}{2\pi\sqrt{\Delta(t)}} e^{-\frac{1}{2\Delta(t)}\left[a(t)\hat{y}^2 - 2h(t)\hat{y}z + b(t)z^2\right]} \\ \Delta(t) = a(t)b(t) - h(t)^2. \end{cases} \tag{55}$$

The marginal TPD $P_M(y,t \mid y_0,0)$ follows from the quadrature:

$$\begin{aligned}P_M(y,t \mid y_0,0) &= \textstyle\int_{\mathbb{R}} e^{-\lambda t}\cosh\left[\sqrt{2\lambda}\ z\right] Q(\hat{y},z,t \mid \hat{y}_0,0,0)dz \\ &= \frac{e^{-\frac{a(t)}{2\Delta(t)}\hat{y}^2} e^{-\lambda t}}{2\pi\sqrt{\Delta(t)}} \left\{\sqrt{\frac{2\pi\Delta(t)}{b(t)}} e^{\frac{h(t)^2\hat{y}^2}{2\Delta(t)b(t)}} \cosh\left[\sqrt{2\lambda}\frac{h(t)}{b(t)}\hat{y}\right] e^{\lambda\frac{\Delta(t)}{b(t)}}\right\} \\ &= \frac{1}{\sqrt{2\pi b(t)}} e^{-\frac{\hat{y}^2}{2b(t)}} \cosh\left[\sqrt{2\lambda}\frac{h(t)}{b(t)}\hat{y}\right] e^{\lambda\left[\frac{\Delta(t)}{b(t)} - t\right]}.\end{aligned} \tag{56}$$

Now provided we have:

$$a(t) = b(t) = h(t) = t \quad \Rightarrow \quad \Delta(t) = a(t)b(t) - h^2(t) = 0, \tag{57}$$

we see that equation (56) can be rewritten as:

$$P_M(y,t \mid y_0,0) = \frac{1}{\sqrt{2\pi t}}\left\{e^{-\frac{(\hat{y}-\sqrt{2\lambda}t)^2}{2t}} + e^{-\frac{(\hat{y}+\sqrt{2\lambda}t)^2}{2t}}\right\}, \tag{58}$$

which follows immediately when one uses the Bernoulli representation equation (23) of the ballistic noise source. The coefficients $a(t)$, $b(t)$ and $h(t)$ given in equation (57) follow by using the Chandrasekhar (1943) general procedure.[5] Let us remark from equation (55) that $\Delta(t) = 0$ indicates that the $(\hat{Y}_t, Z_t)$ is actually a degenerate diffusion process

[5] Here one directly uses Lemma II and equations (260)-(263).

in $\mathbb{R}^2$. This can be understood by observing that the stochastic differential equation corresponding to the diffusion operator (54) reads:

$$\begin{cases} d\hat{Y}_t = dW_t, \\ dZ_t = dW_t. \end{cases} \tag{59}$$

This shows that $\hat{Y}_t = Z_t + \text{const} = W_t$, and the underlying dynamics degenerates to a scalar Wiener process on the $\mathbb{R}^2$ plane.

In terms of the original asset dynamics, X_t reduces to:

$$\begin{aligned} P_M(x,t \mid x_0,0) &= \tfrac{1}{2x\sqrt{2\pi\sigma^2 t}} \left\{ e^{-\frac{[\ln(x)-\ln(x_0)-(\mu-\sigma\sqrt{2\lambda})t]^2}{2\sigma^2 t}} + e^{-\frac{[\ln(x)-\ln(x_0)-(\mu+\sigma\sqrt{2\lambda})t]^2}{2\sigma^2 t}} \right\} \\ &:= \tfrac{1}{2} P_{\mathrm{BS}}^{(-)}(x,t \mid x_0,0) + \tfrac{1}{2} P_{\mathrm{BS}}^{(+)}(x,t \mid x_0,0) \end{aligned} \tag{60}$$

which is the superposition of a pair of Black-Scholes probability densities $P_{\mathrm{BS}}^{(\pm)}(x,t \mid x_0,0)$ with rates $\mu \pm \sigma\sqrt{2\lambda}$. It is worth pointing out that it can be the case that the ballistic component leads to $\mu - \sigma\sqrt{2\lambda} < 0$, thus exhibiting a net drifting tendency to be absorbed in the bankrupt state $x = 0$. Let us calculate the first moment $\mathbb{E}^{(\lambda)}(X_t \mid x_0)$ for the asset dynamics given by (52). To this aim, we first observe that:

$$\begin{aligned} dX_t = (\mu \pm \sigma\sqrt{2\lambda})dt + \sigma dW_t \quad &\Rightarrow \quad \mathbb{E}(X_t \mid x_0) = x_0 e^{(\mu\pm\sigma\sqrt{2\lambda})t}\, \mathbb{E}\left\{e^{\sigma W_t}\right\} \\ &= x_0 e^{(\mu-\sigma\sqrt{2\lambda})t} \textstyle\sum_{n=0}^{\infty} \mathbb{E}\left\{\frac{(\sigma W_t)^n}{n!}\right\} = x_0 e^{\left(\mu\pm\sigma\sqrt{2\lambda}+\sigma^2\right)t}. \end{aligned} \tag{61}$$

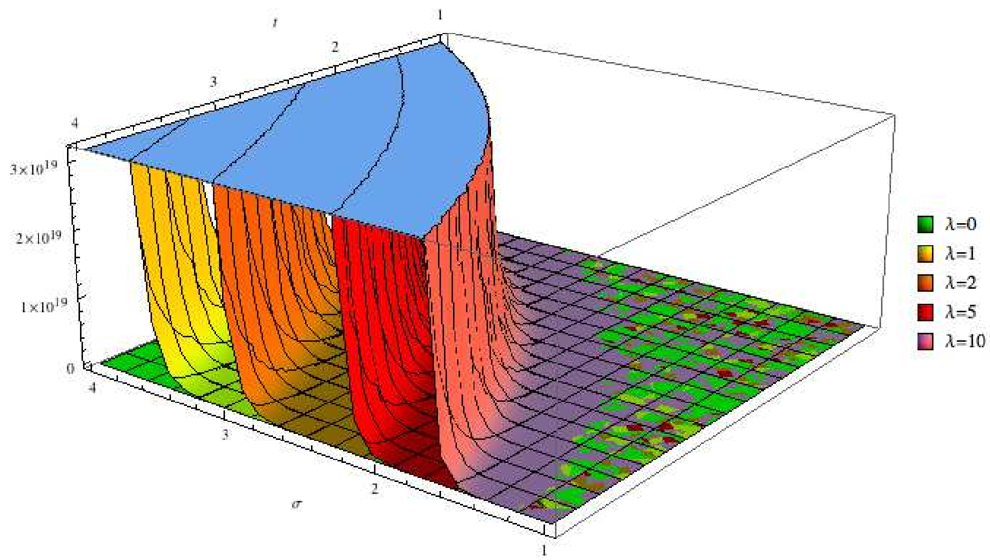

Figure 2: An illustration of the Black-Scholes average asset dynamics $\mathbb{E}^{(\lambda)}(X_t \mid x_0) = x_0 e^{\left(\mu+\sigma^2\right)t} \cosh\left[\sigma\sqrt{2\lambda}t\right]$, for $\lambda = 0, 1, 2, 5, 10$ (we set $x_0 = 1, \mu = 10$).

Accordingly, we have:

$$\mathbb{E}^{(\lambda)}(X_t \mid x_0) = \frac{x_0}{2}\left\{x_0 e^{\left(\mu-\sigma\sqrt{2\lambda}+\sigma^2\right)t} + e^{\left(\mu+\sigma\sqrt{2\lambda}+\sigma^2\right)t}\right\} = x_0 e^{\left(\mu+\sigma^2\right)t}\cosh\left[\sigma\sqrt{2\lambda}t\right]. \quad (62)$$

Comparing the average asset growth for WGN with respect to the ballistic driving noise, one notes that:

$$\frac{\mathbb{E}^{(\lambda)}(X_t \mid x_0)}{\mathbb{E}^{(0)}(X_t \mid x_0)} = \cosh\left[\sigma\sqrt{2\lambda}t\right] \quad (63)$$

and therefore a net average growth enhancement due to the ballistic driving environment. An illustration of equation (62) is provided in Figure 2 for the standard Black-Scholes case ($\lambda = 0$), and progressively higher values of λ.

5 Conclusions

In this paper, we have extended the well-known concept of a mean-preserving spread to diffusion processes. We have shown that ballistic super-diffusive noise, because it is the only Brownian bridge with non-constant drift that satisfies the dynamic equivalent to Rothschild and Stiglitz's (1970) integral conditions, is a leading case that deserves to be in the standard toolkit of economists because it allows one to go beyond normality while preserving analytical tractability.

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